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Finite-amplitude neutral disturbances in plane Poiseuille flow

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Finite-amplitude disturbances in plane Poiseuille flow are studied by a method involving Fourier expansion with numerical solution of the resulting partial differential equations in the coefficient functions. A number of solutions are developed which extend to relatively long times so that asymptotic stability or instability can be established with a degree of confidence. The amplitude for neutral stability is established for a fixed wavenumber for two values of the Reynolds number. Details of the neutral velocity fluctuation are presented. These and earlier results are expressed in terms of the asymptotic amplitude and compared with results obtained by prior workers. The results indicate that the expansion methods used by prior workers may be valid only for amplitudes considerably smaller than 0·01.

1. Introduction

George & Hellums (1972) made a study of the response of plane Poiseuille flow to a particular finite-amplitude disturbance. A number of numerical solutions were developed for a fixed wavenumber for various values of the initial amplitude and Reynolds number. In each case, after observation of the behaviour of the solution, a decision was made as to whether it was stable (amplitude diminishing with time) or unstable. The amplitude ordinarily was not a simple monotone function of time. In some cases, a solution must be observed over a relatively long period of time before the asymptotic behaviour can be established with a degree of confidence. In fact, short-term observation of a solution can lead one to make serious errors on the stability question. The amplitude for neutral stability (the asymptotic amplitude in a case of neutral stability) is often much smaller than the initial amplitude. Furthermore, since the neutral amplitude does not occur until many cycles of the disturbance have taken place, the neutral disturbance presumably bears little influence of the shape of initial disturbance and may be considered to approximate the large-time asymptote.

In this communication we present results of a new series of solutions by George & Hellums' method. These solutions were developed for the primary purpose of examining this asymptote or neutral disturbance in detail.

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To accomplish this purpose, we have carried out a number of solutions to very long times so that we can be relatively confident that we are observing an approximation to the asymptotic behaviour. In addition, we have reinterpreted the work of George & Hellums and developed a summary of the work using the asymptotic amplitude as a parameter of neutral stability. In the original work, the initial amplitude was used as a parameter. As indicated below, this reinterpretation has important implications, especially in comparisons with results of other workers.

Details of the method are given by George (1970) and by George & Hellums (1972). Hence, we confine our remarks here to a very brief outline.

The dimensionless disturbance stream function satisfies

$$\frac{\partial(\nabla^2\psi)}{\partial t}+\frac{\partial(\psi,\nabla^2\psi)}{\partial(x,y)}+(1-y^2)\frac{\partial(\nabla^2\psi)}{\partial x}-2\frac{\partial\psi}{\partial x}=\frac{1}{R}\nabla^4\psi, \tag{1}$$

where R is the Reynolds number based on half the plate separation and the maximum velocity of the undisturbed flow. The no-slip boundary conditions to be satisfied on the channel walls are

$$\psi(\pm 1)=0, \quad \partial\psi(\pm 1)/\partial y=0. \tag{2}$$

Approximate solutions are sought by assuming a truncated form of the expansion

$$\psi(x,y,t)=\sum_{m=0}^{\infty}\{A_m(y,t)\cos m\alpha x+B_m(y,t)\sin m\alpha x\}. \tag{3}$$

The functions $A_m(y,t)$ and $B_m(y,t)$ are found as solutions of the system of coupled partial differential equations that appears when the assumed expansion for ψ is substituted in (3) and like trigonometric terms equated. For complete details of the finite-difference method used to solve the initial-value problems see George (1970).

Following George & Hellums, we take the initial disturbance to be of the form

$$\left.\begin{aligned} A_1(y,0)=k_A f(a,y), \quad A_m(y,0)=0 \quad \text{for} \quad m\neq 1,\\ B_m(y,0)=0 \quad \text{for all} \quad m, \end{aligned}\right\} \tag{4}$$

where $f(a,y)$ is the function

$$\left(\frac{\cosh ay}{\cosh a}-\frac{\cos ay}{\cos a}\right)$$

normalized so that its maximum value at $y=0$ is 1, and where a is a root of the transcendental equation

$$\tanh a+\tan a=0 \tag{5}$$

and k_A is the amplitude.

From linear theory, the odd modes are all strongly stable and so the decision was made to concentrate on even disturbances in these investigations. The form of (4) was retained so that results could be compared with those obtained earlier. However, as will be seen, the shape of the neutral oscillation is so far removed from that of the assumed initial disturbance that, if further work were to be done along these lines, it would be worthwhile starting with a profile which more

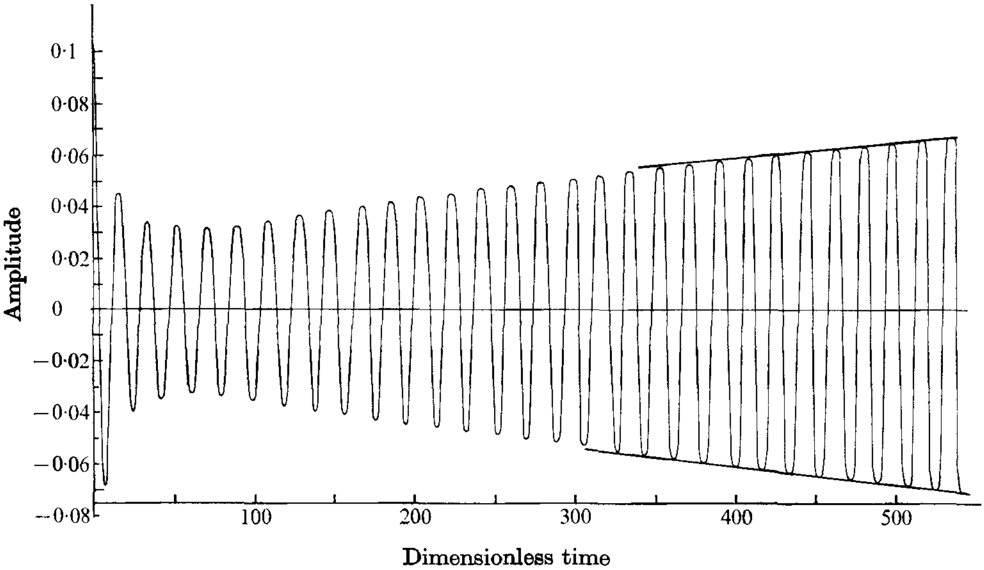


FIGURE 1. Disturbance stream function at $x = 0$ and $y = 0$ with $R = 4000$, $\alpha = 1{\cdot}05$ and $k_A = 0{\cdot}105$.

closely resembled that of the computed neutral flow. The calculations were carried out with $m = 0$, 1 and 2. George & Hellums studied the effect of higher harmonics (up to $m = 4$) and found their effect to be small for the amplitudes to be discussed herein.

To locate a neutral amplitude, assuming that one exists, for a given Reynolds number and wavenumber α we must start with various trial values of k_A and solve an initial-value problem, developing the solutions in time until we are reasonably certain that the amplitude is exhibiting either continuous growth or continuous decay. In this fashion we may, by carefully selecting k_A, close in on the desired constant-amplitude solution.

2. Results and discussion

Figure 1 shows a typical plot of the development of the disturbance stream function on the centre-line at $x = 0$ as a function of dimensionless time t with $R = 4000$, $\alpha = 1{\cdot}05$ and $k_A = 0{\cdot}105$. The motion at first decays very rapidly, as always seems to be the case, at least with this particular form of the initial disturbance, passes through a minimum and then grows with a decreasing growth rate until ultimately appearing to increase linearly. (As will be seen in figure 2 (a), this case was carried to a considerably longer development time than is shown in figure 1, the linear growth still being maintained.)

To compare results for different values of k_A we plot the relative maxima of the modulus of the amplitude as a discrete function of t. This is done in figure 2 (a) for the particular case of $R = 4000$ and $\alpha = 1{\cdot}05$ and it almost certainly establishes that there is a neutral-amplitude solution with k_A falling between $0{\cdot}0950$ and

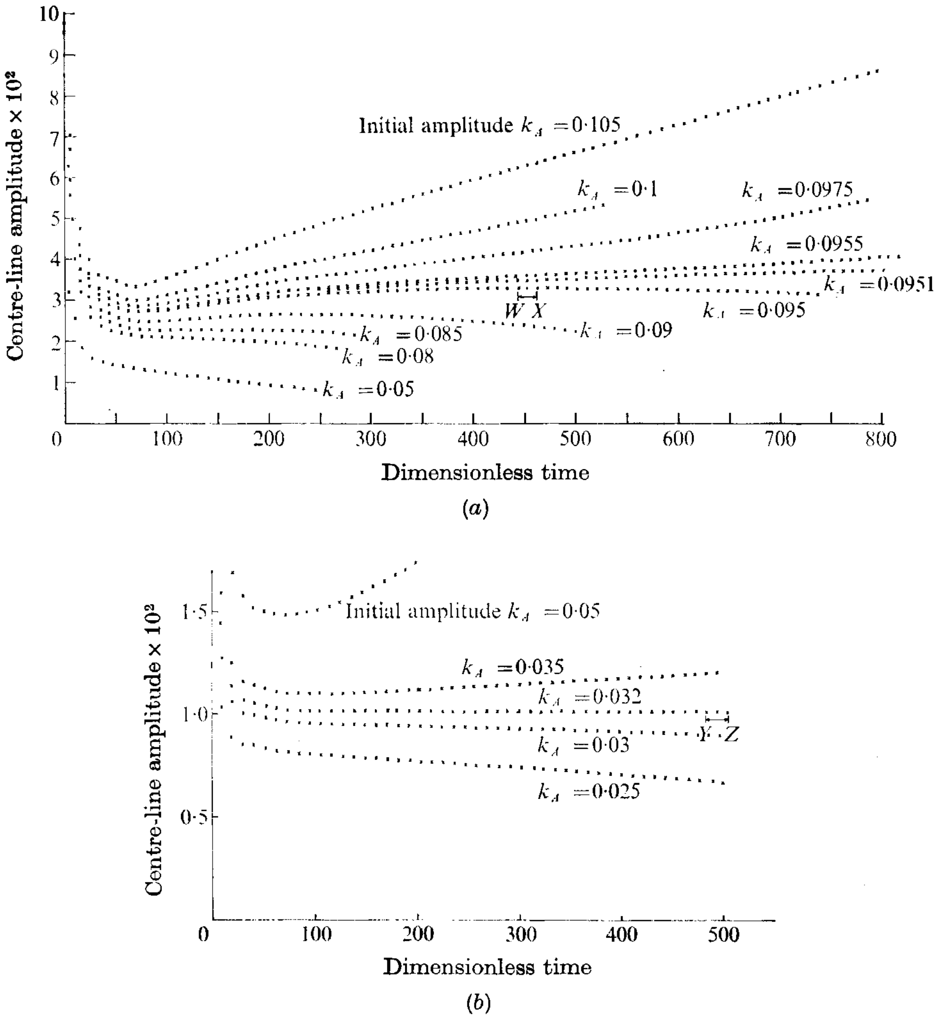
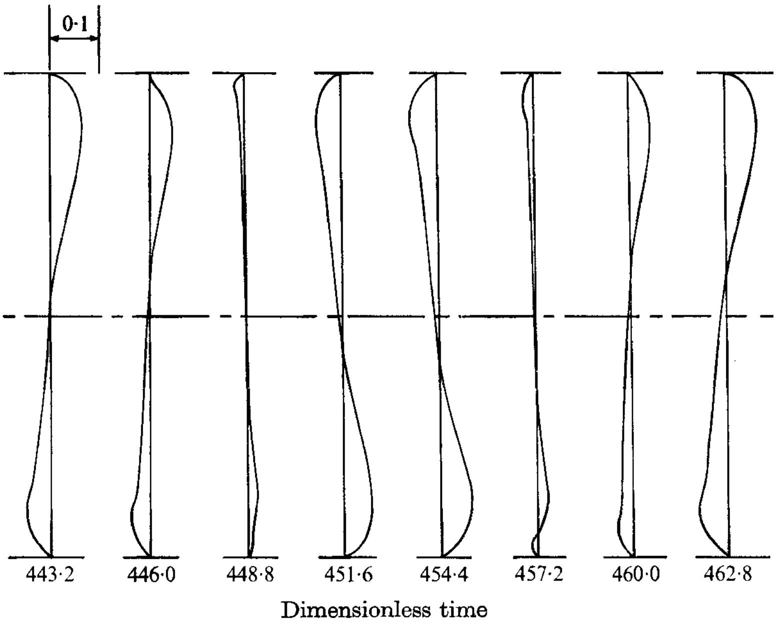


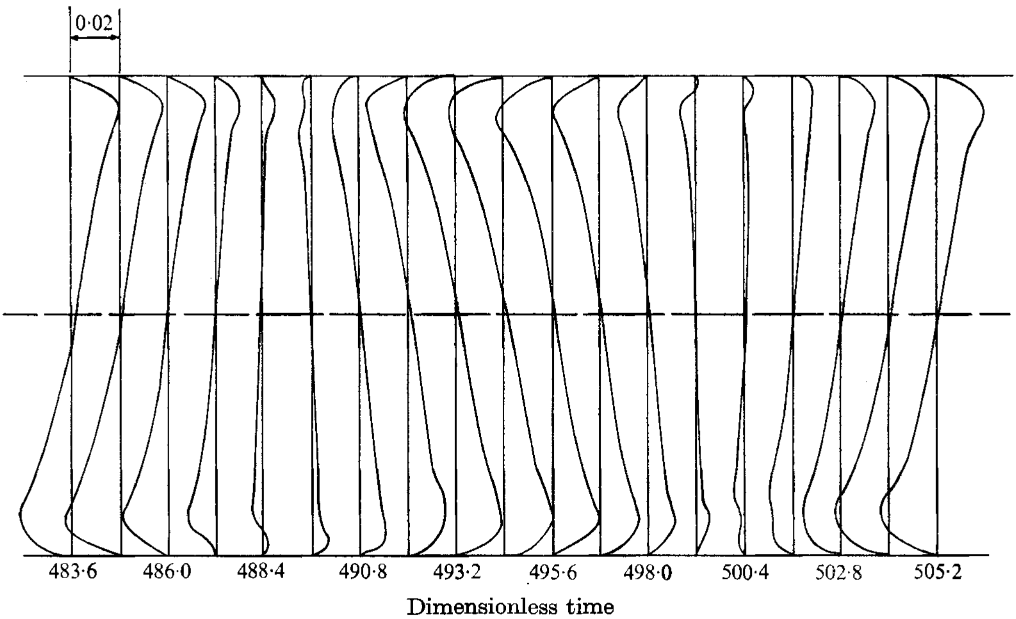
FIGURE 2. Stream function amplitude extrema on the centre-line at $x = 0$ as a function of dimensionless time for different initial amplitudes with $\alpha = 1\cdot05$. (a) $R = 4000$. (b) $R = 5200$.

0·0951. The corresponding neutral amplitude lies in the range 0·033–0·035. In figure 2(b), which shows results of calculations with $R = 5200$ and $\alpha = 1\cdot05$, the neutral amplitude generated with $k_A = 0\cdot032$ is well determined and is found to be 0·0102.

Figures 3(a) and (b), corresponding to the ranges WX and YZ indicated in figures 2(a) and (b) respectively, show the down-channel component of the disturbance velocity as a function of t across the channel at $x = 0$. These profiles are for one complete oscillation of the flow. Note, however, that in both cases the disturbance velocity on the centre-line is one-signed although very small. This effect may be due to the fact that we are close to, but not exactly at the neutral solution and it is conjectured that if the search were further refined this effect would disappear.



(*a*)



(*b*)

FIGURE 3. One complete velocity profile oscillation close to the neutral solution at $x = 0$. (*a*) $R = 4000$, $\alpha = 1{\cdot}05$ and $k_A = 0{\cdot}095$. (*b*) $R = 5200$, $\alpha = 1{\cdot}05$ and $k_A = 0{\cdot}032$.

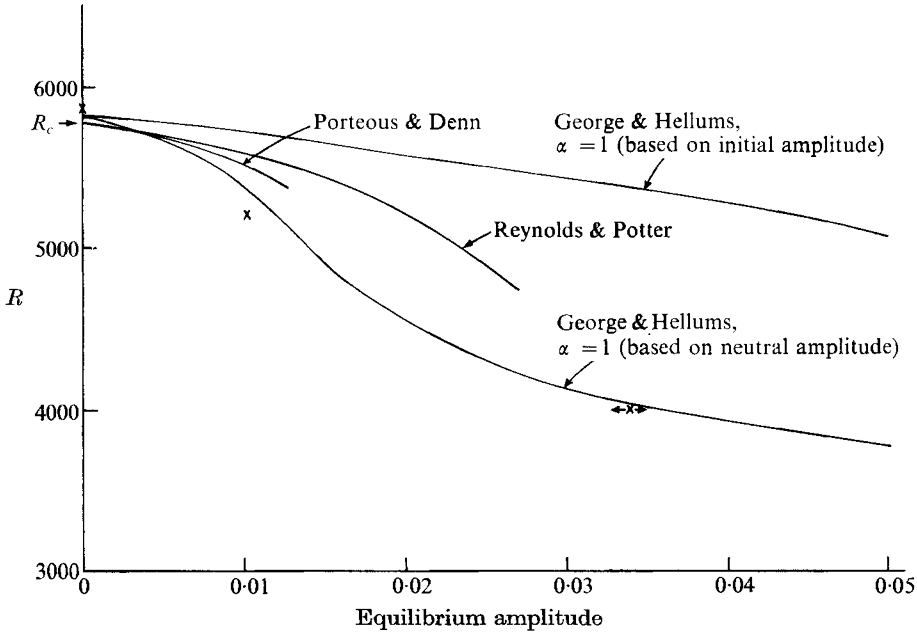


FIGURE 4. Comparison of the calculated neutral amplitudes for $\alpha = 1{\cdot}05$ with the results of previous work.

In George & Hellums (1972) the comparison of results for $\alpha = 1$ with the analytical work of other authors was presented in terms of the initial disturbance amplitude k_A rather than the computed neutral amplitude. Since the neutral amplitudes are considerably lower than k_A, this form of presentation misrepresents the degree of agreement between the different approaches. Also, the previous work carried out with $\alpha = 1$ may be less accurate since the decision relating to growth was made when the solution was less fully developed. As can be seen from figure 2, the estimate of the neutral amplitude can be in error if estimated on the basis of relatively short-time solutions.

In figure 4, the two computed cases are compared with other work on the basis of the neutral amplitudes. Shown on this figure are the small-amplitude curve constructed by Porteous (1971) and that computed by Reynolds & Potter (1967). Note that the latter differs from that reproduced in George & Hellums (1972), thus rectifying the omission of a factor $1/2^{\frac{1}{2}}$ in that paper. The value of R for an infinitesimal disturbance and $\alpha = 1{\cdot}05$ was interpolated from curves given by Porteous. In making comparisons of this type it is important to recognize that the amplitudes in this work are twice those of the various workers who expand in the complex exponential form rather than the real trigonometric form.

Notice that the two points from this work fall well below the analytical results. Of course, they may themselves be well above the envelope of neutral stability. Our results, therefore, suggest that the validity of some expansion methods is restricted to amplitudes considerably less than 0·01.

In figure 4 we also show the earlier work of George & Hellums presented both in terms of the initial amplitude and in terms of the neutral amplitude. It can be seen that the neutral amplitudes differ greatly from the corresponding initial amplitudes, especially in the higher amplitude cases, which are those of most interest. The neutral amplitude is the more satisfying parameter to use in these comparisons since earlier workers have usually used this amplitude in their various analytical approaches. Furthermore, the effect of the initial amplitude would presumably be dependent on the profile of the initial disturbance. The neutral amplitude should be independent of the initial conditions to the degree that we approximate the actual asymptotic behaviour.

Meksyn & Stuart (1951) and Pekeris & Shkoller (1969*a*, *b*) have also applied expansion methods in this problem. Comparisons of these methods have been given by George & Hellums (1972). The second Pekeris & Shkoller work (1969*b*) presents results which agree with Reynolds & Potter at least within the limitations of the scale of figure 4.

Ideally one would like to construct a series of neutral curves of the type shown in figure 4 for various wavenumbers. Thus, the envelope determined by the lowest curve at each amplitude would represent the critical stability curve. As a step in that direction, the new solutions were developed for $\alpha = 1{\cdot}05$ rather than $\alpha = 1{\cdot}00$ since exploratory calculations indicated that this would yield a lower transition Reynolds number. As can be seen in figure 4, these points of the current work are lower than the $\alpha = 1{\cdot}00$ curve. However, this difference should be viewed with some caution, since more care in determining the exact amplitude of neutral stability was taken in the current work. The computer time required to complete the two-parameter search in the determination of the complete critical stability curve would be relatively large. Hence, we have no plans to carry out such extensive calculations. We have presented results for relatively few cases. However, we feel that these results are reliable and they contribute to the understanding of the transition. We also hope that these results will be useful to future workers in assessing the validity of various approximation methods.

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